such that T_n and its first n derivatives have the same values at $x = a$ as f and its first n derivatives. By differentiating repeatedly and setting $x = a$, show that these conditions are satisfied if $c_0 = f(a)$, $c_1 = f'(a)$, $c_2 = \frac{1}{2}f''(a)$, and in general

$$c_k = \frac{f^{(k)}(a)}{k!}$$

where $k! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot k$. The resulting polynomial

$$T_n(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

is called the **nth-degree Taylor polynomial of f centered at a.**

6. Find the eighth-degree Taylor polynomial centered at $a = 0$ for the function $f(x) = \cos x$. Graph f together with the Taylor polynomials T_2, T_4, T_6, T_8 in the viewing rectangle $[-5, 5]$ by $[-1.4, 1.4]$ and comment on how well they approximate f.

3 Review

CONCEPT CHECK

1. State each of the following differentiation rules both in symbols and in words.
(a) The Power Rule
(b) The Constant Multiple Rule
(c) The Sum Rule
(d) The Difference Rule
(e) The Product Rule
(f) The Quotient Rule
(g) The Chain Rule

2. State the derivative of each function.

(a) $y = x^n$	(b) $y = e^x$	(c) $y = a^x$
(d) $y = \ln x$	(e) $y = \log_a x$	(f) $y = \sin x$
(g) $y = \cos x$	(h) $y = \tan x$	(i) $y = \csc x$
(j) $y = \sec x$	(k) $y = \cot x$	(l) $y = \sin^{-1}x$
(m) $y = \cos^{-1}x$	(n) $y = \tan^{-1}x$	(o) $y = \sinh x$
(p) $y = \cosh x$	(q) $y = \tanh x$	(r) $y = \sinh^{-1}x$
(s) $y = \cosh^{-1}x$	(t) $y = \tanh^{-1}x$	

3. (a) How is the number e defined?
(b) Express e as a limit.
(c) Why is the natural exponential function $y = e^x$ used more often in calculus than the other exponential functions $y = a^x$?
(d) Why is the natural logarithmic function $y = \ln x$ used more often in calculus than the other logarithmic functions $y = \log_a x$?

4. (a) Explain how implicit differentiation works.
(b) Explain how logarithmic differentiation works.

5. What are the second and third derivatives of a function f? If f is the position function of an object, how can you interpret f'' and f'''?

6. (a) Write an expression for the linearization of f at a.
(b) If $y = f(x)$, write an expression for the differential dy.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. If f and g are differentiable, then

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

2. If f and g are differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$$

3. If f and g are differentiable, then

$$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$$

4. If f is differentiable, then $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{f'(x)}{2\sqrt{f(x)}}$.

5. If f is differentiable, then $\dfrac{d}{dx}f(\sqrt{x}) = \dfrac{f'(x)}{2\sqrt{x}}$.

6. If $y = e^2$, then $y' = 2e$.

7. $\dfrac{d}{dx}(10^x) = x10^{x-1}$

8. $\dfrac{d}{dx}(\ln 10) = \dfrac{1}{10}$

9. $\dfrac{d}{dx}(\tan^2 x) = \dfrac{d}{dx}(\sec^2 x)$

10. $\dfrac{d}{dx}|x^2 + x| = |2x + 1|$

11. If $g(x) = x^5$, then $\lim_{x\to 2} \dfrac{g(x) - g(2)}{x - 2} = 80$.

12. $\dfrac{d^2y}{dx^2} = \left(\dfrac{dy}{dx}\right)^2$

13. An equation of the tangent line to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

EXERCISES

1–46 □ Calculate y'.

1. $y = (x + 2)^8(x + 3)^6$

2. $y = \sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$

3. $y = \dfrac{x}{\sqrt{9 - 4x}}$

4. $y = \dfrac{e^x}{1 + x^2}$

5. $y = \sin(\cos x)$

6. $y = \sin^{-1}(e^x)$

7. $y = xe^{-1/x}$

8. $y = x^r e^{sx}$

9. $y = \tan\sqrt{1 - x}$

10. $y = \dfrac{1}{\sin(x - \sin x)}$

11. $y = \dfrac{x}{8 - 3x}$

12. $y = \left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}}$

13. $y = \sec 2\theta$

14. $y = -2/\sqrt[4]{t^3}$

15. $y = (1 - x^{-1})^{-1}$

16. $y = \ln(\csc 5x)$

17. $y = e^{cx}(c\sin x - \cos x)$

18. $y = \ln(x^2 e^x)$

19. $y = e^{e^x}$

20. $y = 5^{x\tan x}$

21. $x^2y^3 + 3y^2 = x - 4y$

22. $x\tan y = y - 1$

23. $y = \sqrt[5]{x\tan x}$

24. $y = \sec(1 + x^2)$

25. $x^2 = y(y + 1)$

26. $y = 1/\sqrt[3]{x + \sqrt{x}}$

27. $y = \dfrac{(x - 1)(x - 4)}{(x - 2)(x - 3)}$

28. $y = \sqrt{\sin\sqrt{x}}$

29. $y = \log_{10}(x^2 - x)$

30. $y = e^{\cos x} + \cos(e^x)$

31. $y = \ln\sin x - \frac{1}{2}\sin^2 x$

32. $y = \arctan(\arcsin\sqrt{x})$

33. $y = \sin(\tan\sqrt{1 + x^3})$

34. $xe^y = y - 1$

35. $y = \cot(3x^2 + 5)$

36. $y = \dfrac{(x + \lambda)^4}{x^4 + \lambda^4}$

37. $y = \cos^2(\tan x)$

38. $y = \dfrac{\sin mx}{x}$

39. $y = \dfrac{\sqrt{x + 1}\,(2 - x)^5}{(x + 3)^7}$

40. $y = \ln|\csc 3x + \cot 3x|$

41. $y = x\sinh(x^2)$

42. $y = x^{\cos x}$

43. $y = \ln(\cosh 3x)$

44. $y = \ln\left|\dfrac{x^2 - 4}{2x + 5}\right|$

45. $y = \cosh^{-1}(\sinh x)$

46. $y = x\tanh^{-1}\sqrt{x}$

47. If $f(x) = 1/(2x - 1)^5$, find $f''(0)$.

48. If $g(t) = \csc 2t$, find $g'''(-\pi/8)$.

49. Find y'' if $x^6 + y^6 = 1$.

50. Find $f^{(n)}(x)$ if $f(x) = 1/(2 - x)$.

51. Use mathematical induction to show that if $f(x) = xe^x$, then $f^{(n)}(x) = (x + n)e^x$.

52. Evaluate $\lim_{t\to 0} \dfrac{t^3}{\tan^3 2t}$.

53–57 □ Find an equation of the tangent to the curve at the given point.

53. $y = \dfrac{x}{x^2 - 2}$, $(2, 1)$

54. $\sqrt{x} + \sqrt{y} = 3$, $(4, 1)$

55. $y = \tan x$, $(\pi/3, \sqrt{3})$

56. $y = x\sqrt{1 + x^2}$, $(1, \sqrt{2})$

57. $y = \ln(e^x + e^{2x})$, $(0, \ln 2)$

58. If $f(x) = xe^{\sin x}$, find $f'(x)$. Graph f and f' on the same screen and comment.

59. (a) If $f(x) = x\sqrt{5 - x}$, find $f'(x)$.
(b) Find equations of the tangent lines to the curve $y = x\sqrt{5 - x}$ at the points $(1, 2)$ and $(4, 4)$.
(c) Illustrate part (b) by graphing the curve and tangent lines.
(d) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

60. (a) If $f(x) = 4x - \tan x$, $-\pi/2 < x < \pi/2$, find f' and f''.
(b) Check to see that your answers to part (a) are reasonable by comparing the graphs of f, f', and f''.

61. At what points on the curve $y = \sin x + \cos x$, $0 \leqslant x \leqslant 2\pi$, is the tangent line horizontal?

62. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

63. If $f(x) = (x - a)(x - b)(x - c)$, show that

$$\frac{f'(x)}{f(x)} = \frac{1}{x - a} + \frac{1}{x - b} + \frac{1}{x - c}$$

64. (a) By differentiating the double-angle formula

$$\cos 2x = \cos^2 x - \sin^2 x$$

obtain the double-angle formula for the sine function.

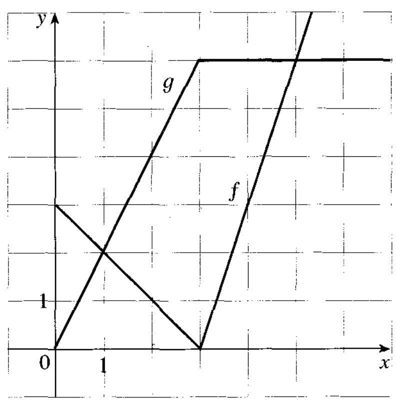
(b) By differentiating the addition formula

$$\sin(x + a) = \sin x \cos a + \cos x \sin a$$

obtain the addition formula for the cosine function.

65. Suppose that $h(x) = f(x)g(x)$ and $F(x) = f(g(x))$, where $f(2) = 3$, $g(2) = 5$, $g'(2) = 4$, $f'(2) = -2$, and $f'(5) = 11$. Find (a) $h'(2)$ and (b) $F'(2)$.

66. If f and g are the functions whose graphs are shown, let $P(x) = f(x)g(x)$, $Q(x) = f(x)/g(x)$, and $C(x) = f(g(x))$. Find (a) $P'(2)$, (b) $Q'(2)$, and (c) $C'(2)$.



67–74 □ Find f' in terms of g'.

67. $f(x) = x^2 g(x)$

68. $f(x) = g(x^2)$

69. $f(x) = [g(x)]^2$

70. $f(x) = g(g(x))$

71. $f(x) = g(e^x)$

72. $f(x) = e^{g(x)}$

73. $f(x) = \ln |g(x)|$

74. $f(x) = g(\ln x)$

75–77 □ Find h' in terms of f' and g'.

75. $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)}$

76. $h(x) = \sqrt{\dfrac{f(x)}{g(x)}}$

77. $h(x) = f(g(\sin 4x))$

78. (a) Graph the function $f(x) = x - 2\sin x$ in the viewing rectangle $[0, 8]$ by $[-2, 8]$.
(b) On which interval is the average rate of change larger: $[1, 2]$ or $[2, 3]$?
(c) At which value of x is the instantaneous rate of change larger: $x = 2$ or $x = 5$?
(d) Check your visual estimates in part (c) by computing $f'(x)$ and comparing the numerical values of $f'(2)$ and $f'(5)$.

79. At what point on the curve $y = [\ln(x + 4)]^2$ is the tangent horizontal?

80. (a) Find an equation of the tangent to the curve $y = e^x$ that is parallel to the line $x - 4y = 1$.
(b) Find an equation of the tangent to the curve $y = e^x$ that passes through the origin.

81. Find a parabola $y = ax^2 + bx + c$ that passes through the point $(1, 4)$ and whose tangent lines at $x = -1$ and $x = 5$ have slopes 6 and -2, respectively.

82. The function $C(t) = K(e^{-at} - e^{-bt})$, where a, b, and K are positive constants and $b > a$, is used to model the concentration at time t of a drug injected into the bloodstream.
(a) Show that $\lim_{t\to\infty} C(t) = 0$.
(b) Find $C'(t)$, the rate at which the drug is cleared from circulation.
(c) When is this rate equal to 0?

83. An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

84. A particle moves along a horizontal line so that its coordinate at time t is $x = \sqrt{b^2 + c^2t^2}$, $t \geq 0$, where b and c are positive constants.
(a) Find the velocity and acceleration functions.
(b) Show that the particle always moves in the positive direction.

85. A particle moves on a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3$, $t \geq 0$.
(a) Find the velocity and acceleration functions.
(b) When is the particle moving upward and when is it moving downward?
(c) Find the distance that the particle travels in the time interval $0 \leq t \leq 3$.

86. The volume of a right circular cone is $V = \pi r^2 h/3$, where r is the radius of the base and h is the height.
(a) Find the rate of change of the volume with respect to the height if the radius is constant.
(b) Find the rate of change of the volume with respect to the radius if the height is constant.

87. The mass of part of a wire is $x(1 + \sqrt{x})$ kilograms, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 4$ m.

88. The cost, in dollars, of producing x units of a certain commodity is

$$C(x) = 920 + 2x - 0.02x^2 + 0.00007x^3$$

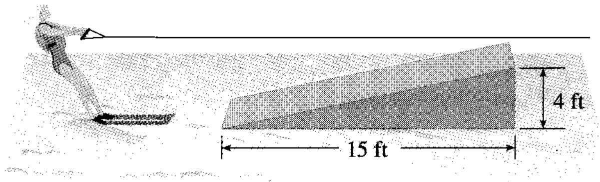
(a) Find the marginal cost function.
(b) Find $C'(100)$ and explain its meaning.
(c) Compare $C'(100)$ with the cost of producing the 101st item.

89. The volume of a cube is increasing at a rate of 10 cm³/min. How fast is the surface area increasing when the length of an edge is 30 cm?

90. A paper cup has the shape of a cone with height 10 cm and radius 3 cm (at the top). If water is poured into the cup at a rate of 2 cm³/s, how fast is the water level rising when the water is 5 cm deep?

91. A balloon is rising at a constant speed of 5 ft/s. A boy is cycling along a straight road at a speed of 15 ft/s. When he passes under the balloon it is 45 ft above him. How fast is the distance between the boy and the balloon increasing 3 s later?

92. A waterskier skis over the ramp shown in the figure at a speed of 30 ft/s. How fast is she rising as she leaves the ramp?



93. The angle of elevation of the sun is decreasing at a rate of 0.25 rad/h. How fast is the shadow cast by a 400-ft-tall building increasing when the angle of elevation of the sun is $\pi/6$?

94. (a) Find the linear approximation to $f(x) = \sqrt{25 - x^2}$ near 3.
(b) Illustrate part (a) by graphing f and the linear approximation.
(c) For what values of x is the linear approximation accurate to within 0.1?

95. (a) Find the linearization of $f(x) = \sqrt[3]{1 + 3x}$ at $a = 0$. State the corresponding linear approximation and use it to give an approximate value for $\sqrt[3]{1.03}$.
(b) Determine the values of x for which the linear approximation given in part (a) is accurate to within 0.1.

96. Evaluate dy if $y = x^3 - 2x^2 + 1$, $x = 2$, and $dx = 0.2$.

97. A window has the shape of a square surmounted by a semicircle. The base of the window is measured as having width 60 cm with a possible error in measurement of 0.1 cm. Use differentials to estimate the maximum error possible in computing the area of the window.

98–100 □ Express the limit as a derivative and evaluate.

98. $\displaystyle\lim_{x \to 1} \frac{x^{17} - 1}{x - 1}$

99. $\displaystyle\lim_{h \to 0} \frac{\sqrt[4]{16 + h} - 2}{h}$

100. $\displaystyle\lim_{\theta \to \pi/3} \frac{\cos\theta - 0.5}{\theta - \pi/3}$

101. Evaluate $\displaystyle\lim_{x \to 0} \frac{\sqrt{1 + \tan x} - \sqrt{1 + \sin x}}{x^3}$.

102. Suppose f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + [f(x)]^2$. Show that $g'(x) = 1/(1 + x^2)$.

103. Find $f'(x)$ if it is known that

$$\frac{d}{dx}[f(2x)] = x^2$$

104. Show that the length of the portion of any tangent line to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ cut off by the coordinate axes is constant.